

Sequential compactness

This note describes an equivalent notion of compactness for metric spaces. It may be useful for this week's homework assignment.

Definition 1. Let (X, d) be a metric space. (X, d) is said to be sequentially compact if for any sequence $\{s_n\}$ in X , there exists a convergent subsequence $s_{n_k} \rightarrow s \in X$.

Theorem 1. Let (X, d) be a metric space. (X, d) is compact if and only if it is sequentially compact.

*Proof.*¹ Suppose that (X, d) is compact and $\{s_n\}$ is a sequence in X with no convergent subsequence. For each $x \in X$, there is an $\epsilon_x > 0$ such that $B(x; \epsilon_x)$ contains only finitely many terms of $\{s_n\}$. (If this were not the case, then one could easily construct a subsequence converging to x .) Varying x , we get an open cover of X by the $B(x; \epsilon_x)$. There must be a finite subcover $B(x_1; \epsilon_{x_1}), \dots, B(x_k; \epsilon_{x_k})$ by compactness. Each $B(x_i; \epsilon_{x_i})$ contains only finitely many terms of $\{s_n\}$, so

$$X = \bigcup_{i=1}^k B(x_i; \epsilon_{x_i})$$

contains only finitely many terms of $\{s_n\}$. Sequences are infinite, so no such $\{s_n\}$ exists and (X, d) is sequentially compact.

Suppose now that (X, d) is sequentially compact. If $\{s_n\}$ is a Cauchy sequence in X , then the convergence $s_{n_k} \rightarrow s$ implies that $s_n \rightarrow s$ by the triangle inequality. Thus (X, d) is complete. Next, consider $\epsilon > 0$ and the open cover of X by $B(x; \epsilon)$ where x varies over all $x \in X$. If this cover does not have a finite subcover, we may inductively find a sequence s_n such that $d(s_k, s_j) > \epsilon$ whenever $k \neq j$. Thus every subsequence of $\{s_n\}$ would fail to be Cauchy and hence would be divergent. By sequential compactness, we may therefore conclude that X is covered by finitely many ϵ -balls.² Starting with $\epsilon = 1$, we may list finitely many points $x_1, \dots, x_{k_1}$ such that every point is within 1 of one of $x_1, \dots, x_{k_1}$. Similarly with $\epsilon = 1/2$, we may list finitely many points $x_{k_1+1}, \dots, x_{k_2}$ such that every point is within $1/2$ of one of $x_{k_1+1}, \dots, x_{k_2}$. Continuing for $\epsilon = 1/3, 1/4, \dots$, we may construct a sequence of points $\{x_n\}$ dense in X .

¹Based on Bredon's Topology and Geometry, although he probably got it from someone else. I think there is a gap in his proof, so there are minor alterations made here.

²Such a space (X, d) is called totally bounded.

Let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of X . Let A have a fixed (arbitrary) well-ordering. This simply means that for every subset B of A , there is a preferred “least” element of B .³ We wish to use $\{x_n\}$ to help count subcovers, so let $V_{n,i}$ denote U_α containing $B(x_i; 1/n)$ with α the least possible choice, whenever such a U_α exists. For each $x \in X$, $\exists \alpha \in A$ such that $x \in U_\alpha$. We may assume that given x , we have chosen the least possible α such that $x \in U_\alpha$. By openness, $B(x; 2/n) \subset U_\alpha$ for some n . We may also find a x_i in our dense sequence such that $d(x, x_i) < 1/n$. Therefore

$$x \in B(x_i; 1/n) \subset B(x; 2/n) \subset U_\alpha.$$

If $\beta < \alpha$, then $x \notin U_\beta$ and thus $B(x_i; 1/n) \not\subset U_\beta$. Therefore $U_\alpha = V_{n,i}$. The $V_{n,i}$ thus give a countable subcover $\{U_{\alpha_n}\}_{n \in \mathbb{N}}$ of $\{U_\alpha\}_{\alpha \in A}$. Let $W_i = U_{\alpha_i}$. If $W_1, W_2, \dots$ has a finite subcover, then we are done. Otherwise, let

$$C_1 = X \setminus W_1, C_2 = X \setminus (W_1 \cup W_2), \dots, C_k = X \setminus \bigcup_{i=1}^k W_i, \dots$$

We have $C_1 \supset C_2 \supset \dots \supset C_k \supset \dots$ and each C_k is nonempty by the lack of a finite subcover. For each n , choose $s_n \in C_n$. There is a convergent subsequence $s_{n_k} \rightarrow s$ by sequential compactness. For each k , $s_{n_l} \in C_{n_l} \subset C_{n_k}$ whenever $l \geq n$, so $s \in C_{n_k}$ as C_{n_k} is closed. k was arbitrary and $n_k \rightarrow \infty$ as $k \rightarrow \infty$, so

$$s \in \bigcap_{k=1}^{\infty} C_{n_k} = X \setminus \bigcup_{k=1}^{\infty} W_{n_k} = \emptyset.$$

This contradiction implies that X is in fact compact. □

With a little more work, one can show that (X, d) being compact is also equivalent to it being complete and totally bounded.

³The assumption that we can choose such an ordering is equivalent to the Axiom of Choice, which we will use freely later anyway.