

Sistemi lineari

1. Norme:

$$\rho(A) = \lambda_M(A) = \max\{|\lambda| \mid \lambda \text{ autoval } A\}$$

$$\|A\|_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^m |a_{ij}|$$

$$\|A\|_1 = \|{}^t A\|_\infty$$

$$\|A\|_2 = \sqrt{\lambda_M({}^t A A)}$$

$$\|A^{-1}\|_2 = \frac{1}{\sqrt{\lambda_m({}^t A A)}}$$

$$A = {}^t A \Rightarrow \|A\|_2 = |\lambda_M(A)|$$

$$A = {}^t A \Rightarrow \|A^{-1}\|_2 = \frac{1}{|\lambda_m(A)|}$$

$$\rho(A) = \inf_{p,q \in \mathbb{N}^*} \|A\|_{p,q}$$

2. Triangolare superiore:

$$x_i = \frac{b_i - \sum_{j=i+1}^n a_{ij} x_j}{a_{ii}} \quad \forall i = n, \dots, 1$$

Triangolare inferiore:

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j}{a_{ii}} \quad \forall i = 1, \dots, n$$

3. Condizionamento:

$$A(x + \Delta x) = b + \Delta b$$

$$\frac{\|\Delta x\|}{\|x\|} \leq \mu(A) \frac{\|\Delta b\|}{\|b\|}$$

$$\mu_2(A) = \frac{\sqrt{\lambda_M({}^t A A)}}{\sqrt{\lambda_m({}^t A A)}}$$

$$A = {}^t A \Rightarrow \mu_2(A) = \frac{|\lambda_M|}{|\lambda_m|}$$

4. Gauss (pivot):

pivot_riga k :

$$\text{Sia } \bar{i} \text{ t.c. } |a_{\bar{i}k}| := \max_{i=k, \dots, n} |a_{ik}|$$

scambio_righe $\bar{i}, k$

pivot_scalato k :

$$d_i := \frac{|a_{ik}|}{\max_{j=k, \dots, n} |a_{ij}|}$$

$$\text{Sia } \bar{i} \text{ t.c. } d_{\bar{i}} = \max_{i=1, \dots, n} d_i$$

$$\forall k = 1, \dots, n-1$$

pivot k

$$\forall i = k+1, \dots, n$$

$$a_{ik} := \frac{a_{ik}}{a_{kk}}$$

$$\forall j = k+1, \dots, n+1$$

$$a_{ij} := a_{ij} - a_{ik} a_{kj}$$

5. Fattorizzazione Cholesky

$A \in \mathbb{R}^{n \times n}$, simmetrica, definita pos.

$A = R^t R$, $R \in \mathbb{R}^{n \times n}$ triang. inf.

$$a_{ij} = \sum_{k=1}^{\min\{i,j\}} R_{ik} R_{jk} \quad \forall i, j$$

$$\forall j = 1, \dots, n$$

$$r_{jj} := \sqrt{a_{jj} - \sum_{k=1}^{j-1} r_{jk}^2}$$

$$\forall i = j+1, \dots, n$$

$$r_{ij} := \frac{a_{ij} - \sum_{k=1}^{j-1} r_{ik} r_{jk}}{r_{jj}}$$

6. Gershgorin-Hadamard

$$A \in \mathbb{C}^{n \times n}$$

$$D_i = \{z \in \mathbb{C} \mid |z - a_{ii}| \leq R_i\}$$

$$R_i = \sum_{j \neq i} |a_{ij}|$$

$$\{\text{autoval. di } A\} \subseteq \bigcup_{i=1}^n D_i$$

7. Metodo delle potenze:

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$

$$x_0 \in \mathbb{C}^{n,1}, \quad x_0 \neq 0$$

$$y_k = \frac{x_k}{\|x_k\|}, \quad x_{k+1} = A y_k$$

$$\sigma_k = \frac{{}^t x_k A x_k}{{}^t x_k x_k} = \frac{{}^t y_k x_{k+1}}{{}^t y_k y_k}$$

$$z_0 = x_0, \quad z_{k+1} = A z_k, \quad \sigma'_k = \frac{{}^t z_k A z_k}{{}^t z_k z_k}$$

$$\lim_{k \rightarrow \infty} \sigma_k = \lim_{k \rightarrow \infty} \sigma'_k = \lambda_1$$

$$\text{Crit. arresto: } \frac{\|(A - \sigma_k I) y_k\|}{\sigma_k \|y_k\|} < \varepsilon$$

$$\text{oppure } |\sigma_{k+1} - \sigma_k| < \varepsilon$$

Interpolazione

1. Lagrange:

$$\lambda_j(x) = \prod_{0 \leq i \leq n, i \neq j} \frac{x - x_i}{x_j - x_i}$$

$$p(x) = \sum_{j=0}^n y_j \lambda_j(x)$$

$$\sum_{j=0}^n \lambda_j(x) (x_j - x)^k \equiv \begin{cases} 1 & k=0 \\ 0 & k>0 \end{cases}$$

$$\lambda_j(x_i) = \delta_{ij}$$

2. Differenze divise:

$$f[x_0] = f(x_0)$$

$$f[x_0, \dots, x_j] =$$

$$= \frac{f[x_1, \dots, x_j] - f[x_0, \dots, x_{j-1}]}{x_j - x_0}$$

$$f[x_0] = f(x_0)$$

$$f[x_1] = f(x_1)$$

$$f[x_2] = f(x_2)$$

$$f[x_0, x_1]$$

$$f[x_1, x_2]$$

$$f[x_0, x_1, x_2]$$

3. Newton:

$$w_0(x) = 1, \quad w_j(x) = \prod_{i=0}^{j-1} (x - x_i)$$

$$p_n(x) = \sum_{i=0}^n f[x_0, \dots, x_i] w_i(x)$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \omega_{n+1}(x)$$

4. Chebyshev

$$t_n(x) = \cos n\varphi(x) : [-1, 1] \longrightarrow \mathbb{R}$$

$$\varphi(x) = \arccos x : [-1, 1] \longrightarrow [0, \pi]$$

$$t_0(x) = 1, \quad t_1(x) = x$$

$$t_n(x) = 2t_{n-1}(x)x - t_{n-2}(x) \quad \forall n > 1$$

$$\deg t_n(x) = n \quad \forall n \geq 0$$

$$c_n = 2^{n-1} \quad \forall n \geq 1$$

radici di t_n :

$$\alpha_k = \cos\left(\frac{\pi}{2n} + k\frac{\pi}{n}\right) \quad k = 0, \dots, n-1$$

$$\text{max: } M_k = \cos\left(k\frac{2\pi}{n}\right) \quad k = 0, \dots, n$$

$$\text{min: } m_k = \cos\left(k\frac{2\pi}{n} + \frac{\pi}{n}\right) \quad k = 0, \dots, n$$

5. Polinomio di Bernstein:

$$B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$$\sum_{k=0}^n B_{n,k}(x) = 1$$

$$B_n(f)(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{n,k}(x)$$

$$\lim_{n \rightarrow +\infty} B_n(f) \stackrel{u.}{=} f \text{ in } [0, 1]$$

$$\text{dove } f \in C^0([0, 1])$$

6. Minimi quadrati

$$\varphi(x, \mathbf{a}) = \sum_{j=1}^m a_j \varphi_j(x), \quad \phi \in \mathbb{R}^{n \times m}$$

$$\phi_{jk} = \varphi_k(x_j), \quad {}^t \phi \phi \mathbf{a} = {}^t \phi \mathbf{y}, \quad \det {}^t \phi \phi \neq 0$$

$$\varepsilon(\mathbf{v})_i = y_i - \varphi(x_i, \mathbf{v}), \quad \|\varepsilon(\mathbf{a})\|_2^2 \text{ minima}$$

Zeri e punti fissi

1.

$$\text{Newton: } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Secanti:

$$x_{k+1} = x_k - f(x_k) \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

Bisezione:

While :

$$c := \frac{a+b}{2}$$

$$\text{Se } |b-a| < 2\varepsilon \vee f(c) = 0 :$$

break

$$\text{Se } f(c)f(a) > 0 : \quad a := c$$

$$\text{Se } f(c)f(a) < 0 : \quad b := c$$

$$\alpha := c$$

2. Criterio di arresto di Aitken per successioni lineari:

$$\text{Hp: } x_n = \varphi(x_{n-1}) \rightarrow \alpha = \varphi(\alpha)$$

$$\text{Hp: } \varphi'(\alpha) \neq 0, x_n \neq \alpha \quad \forall n$$

$$\left| x_n - \frac{x_{n+1}x_{n-1} - x_n^2}{x_{n+1} + x_{n-1} - 2x_n} \right| < \varepsilon$$

3. Minimo di f unimodale:

While

$$x_1 := a + (1-r)(b-a)$$

$$x_2 := a + r(b-a)$$

$$\text{Se } f(x_1) > f(x_2) : \quad a := x_1$$

$$\text{Altrimenti:} \quad b := x_2$$

$$\text{Se } b-a < \varepsilon :$$

$$x_* := a$$

break

Integrali

1. Norma

$$\|I\| = \int_a^b w(x) dx, \quad \|I_n\| = \sum_{i=0}^n |w_i|$$

2. Formula interpolatoria I_n :

$$\deg I_n \geq n$$

$$|I(f) - I_n(f)| \leq \|f - L_n\| \int_a^b w(x) dx$$

$$w_j = \int_a^b w(x) \lambda_j(x) dx \quad \forall j = 0, \dots, n$$

Sistema per il calcolo dei pesi:

$$\int_a^b w(x) x^i dx = \sum_{j=0}^n w_j x_j^i \quad \forall i = 0, \dots, n$$

3. Cambio di pesi e nodi:

$$x_j, w_j \text{ su } [\alpha, \beta], \quad x'_j, w'_j \text{ su } [a, b]$$

$$w'_j = w_j \frac{b-a}{\beta-\alpha}$$

$$x'_j = \frac{b-a}{\beta-\alpha} (x_j - \alpha) + a$$

4. Newton-Cotes: formula interpolatoria su $n+1$ nodi uniformi.

Chiusa

$$x_0 = a, \quad x_n = b, \quad h = \frac{b-a}{n},$$

$$x_i = a + ih \quad i = 0, \dots, n$$

Aperta

$$h = \frac{b-a}{n+2}, \quad x_i = a + (i+1)h \quad \forall i = 0, \dots, n$$

$$\deg I_n = \begin{cases} n & n \text{ dispari} \\ n+1 & n \text{ pari} \end{cases}$$

Resto: n	Chiusa	Aperta
0		$\frac{1}{3} h^3 f^{(2)}(\xi)$
1	$-\frac{1}{12} h^3 f^{(2)}(\xi)$	$\frac{3}{4} h^3 f^{(2)}(\xi)$
2	$-\frac{1}{90} h^5 f^{(4)}(\xi)$	$\frac{14}{45} h^5 f^{(4)}(\xi)$
3	$-\frac{3}{80} h^5 f^{(4)}(\xi)$	$\frac{95}{144} h^5 f^{(4)}(\xi)$
4	$-\frac{8}{945} h^7 f^{(6)}(\xi)$	$\frac{41}{140} h^7 f^{(6)}(\xi)$

5. Formula dei trapezi

$$T(f) \stackrel{\text{def}}{=} \frac{f(a) + f(b)}{2} (b-a)$$

$$w(x) \equiv 1, \quad \deg T = 1$$

$$\text{Resto: } -\frac{(b-a)^3}{12} f''(\xi), \quad \xi \in [a, b]$$

Composita, con distribuzione uniforme dei nodi:

$$C_N(f) = h \left(\frac{f(a) + f(b)}{2} + \sum_{i=1}^{N-1} f(a+ih) \right)$$

$$x_i - x_{i-1} = h = \frac{b-a}{N}$$

$$\text{Resto: } -\frac{b-a}{12} h^2 f''(\xi)$$

Resto di Eulero-MacLaurin:

$$c_1 h^2 + c_2 h^4 + \dots + c_k h^{2k} + \dots$$

6. Cavalieri-Simpson (formula interpolatoria)

$$\text{Nodi } x_0 = a, x_1 = \frac{a+b}{2}, x_2 = b$$

$$S(f) = \frac{b-a}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

$$w(x) \equiv 1, \quad \deg S = 2$$

$$\text{Resto formula composita: } -\frac{b-a}{2880} h^4 f^{(4)}(\xi)$$

$$\text{con } x_i - x_{i-1} = h = \frac{b-a}{N}$$

7. Gauss-Chebyshev:

$$w(x) \equiv \frac{1}{\sqrt{1-x^2}}, \quad [a, b] = [-1, 1], \quad \deg I_n = 2n+1$$

$$w_i = \frac{\pi}{n+1} \quad \forall i = 0, \dots, n$$

$$x_i = \cos \left[\frac{1}{n+1} \left(\frac{\pi}{2} + i\pi \right) \right] \quad i = 0, \dots, n$$

8. Estrapolazione di Richardson:

$$F_1(f, h) = I_n(f, h), \quad h < 1$$

$$F_{k+1}(f, h) = \frac{q^{pk} F_k(f, h) - F_k(f, qh)}{q^{pk} - 1}$$

Per la formula dei trapezi: $p_k = 2k$

9. Integrazione adattiva:

IntAdattiva(α, β) :

$$M =: I_n(\alpha, \beta)$$

$$N =: I'_n(\alpha, \beta)$$

$$\text{Se } |M - N| < \frac{\varepsilon}{b-a} (\beta - \alpha)$$

$$I =: N$$

Altrimenti :

$$I =: \text{IntAdattiva} \left(\alpha, \frac{\alpha+\beta}{2} \right) + \text{IntAdattiva} \left(\frac{\alpha+\beta}{2}, \beta \right)$$

$$I =: \text{IntAdattiva}(a, b)$$